A Geometric Proposition.

By E. Lasker.

The elementary proposition that, "if the corners $(A, B, C)(D, E, F)$ of two triangles in a plane are such that AD, BE, CF are collinear, then the reciprocal property is true of the sides respectively opposite to the points," admits of a very wide extension that may not be without some importance. The fact, which is going to be explained, has applications to the geometry of a space of n manifoldness; but, to simplify the diction, it will be described as applying only to geometry in a plane or space. It may be announced in this fashion:

Proposition. Let A, B, C, D, E, F be six points in a plane, or A, B, C, D, E, F, G, H be eight points in space. Let Ω be any configuration of these points, which is characterized by the vanishing of one or a set of linear invariants i of above set of points respectively. In that case, any set of points forming Ω gives rise, by separating the points into two triangles or tetrahedra, to six sides or eight planes which will always again form Ω.

To be quite accurate, we add that if A, B, C, D, E, F form Ω, and the two triangles, into which the six points are divided, are $(A, B, C)(D, E, F)$, then the lines BC, CA, AB, DE, EF, FD form the reciprocal Ω. And similarly in space.

The demonstration of the proposition is thus: An invariant i linear in the coefficients of A, B, C, D, E, F is of the shape

$$c_1 (ABC).(DEF) + c_2 (ABD).(CEF) + c_3 (ADE).(BCF) + \ldots\ldots,$$

where the $c_1, c_2, c_3 \ldots\ldots$ are numerical constants. Replacing in above expression A by BC, B by CA, C by AB, D by EF, E by FD, F by DE and form-

ing the reciprocal invariant i', it is found by elementary properties of determinants that

$$i' = (ABC).(DEF).i.$$

Hence, if $i = 0$, also $i' = 0$, and if a set of invariants $i = 0$, then also the corresponding set $i' = 0$. Q. E. D.

The demonstration of the proposition for higher spaces is analogous.

To give a few easy applications: If eight points A, B, C, D, E, F, G, H in space are such that the line common to the planes ABC and DEF intersects GH, then will the line common to ADE and BCF, the point of intersection of DEG and FH, and that of BCH and AG lie in one plane. Or else: If the quadric through the lines AB, CD, EF admits G, H as conjugate points, then will the quadric through CD, AB, GH admit EFH, EFG as conjugate planes; and the quadric through CG, DH and the line common to ABG and EFH will admit ABC and DEF as conjugate planes; and the quadric through CGH/DEF (read: the line common to CGH and DEF), AGH/BEF and BD will admit ACG and ACH as conjugate planes.

It is, of course, possible to apply the proposition to the linear invariants of $2m$ elements of a m-fold linear manifoldness. As an instance, let the manifoldness in question be that of the conics in a given plane whose m is 6. Let

$$u_1, u_2 \ldots . u_{12}$$

be twelve conics, and let the vanishing of the invariant i signify that there is a conic belonging to all three involutions

$$(u_1, u_2, u_3, u_4), \ (u_5, u_6, u_7, u_8) \text{ and } (u_9, u_{10}, u_{11}, u_{12}).$$

Interpreting the $u_1 \ldots . u_{12}$ as squares of lines, the proposition means this: "If three quadruples (l_1, l_2, l_3, l_4), $(l_5, l_6 \ldots .)(l_9 \ldots .)$ of lines are said to be in relation Ω whenever three conics exist touching the respective quadruples and having in common four tangents, then any system of lines $l_1 \ldots . l_{12}$ in relation Ω determines systems of points $p_1 \ldots . p_{12}$ also in relation Ω. The points $p_1 \ldots p_{12}$ are found in this fashion: Any two quadruples of lines $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\lambda_5 \ldots . \lambda_8)$ always uniquely determine a quadruple of points $(\pi_1, \pi_2, \pi_3, \pi_4)$, namely, those four points whose squares form the linear involution comprising

the systems of conics that touch the quadruples of lines respectively. Thus,

	(p_1, p_2, p_3, p_4)
are determined by	(l_5, l_6, l_9, l_{10})
and	$(l_7, l_8, l_{11}, l_{12})$,
p_5, p_6, p_7, p_8 by	$(l_9, l_{10}, l_{11}, l_2)$
and	$(l_{11}, l_{12}, l_3, l_4)$,
$p_9, p_{10}, p_{11}, p_{12}$ by	(l_1, l_2, l_5, l_6)
and	(l_3, l_4, l_7, l_8).

These applications will have given a sufficient insight into the extent as well as the limitations of the proposition.

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